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Renormalization group and infinite algebraic structure in D-dimensional conformal field theory

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Abstract
We consider scalar field theory in the D-dimensional space with nontrivial metric and local action functional of the most general form. For this model it is possible to construct a generalization of the renormalization procedure and the RG-equations. In the fixed point the diffeomorphism and Weyl transformations generate an infinite algebraic structure of D-dimensional conformal field theory models. The Wilson expansion and crossing symmetry enable us to obtain sum rules for dimensions of composite operators and Wilson coefficients.

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1. Introduction

Essential achievements in the study of quantum field theoretical models were obtained on the basis of the analysis of their symmetry properties and algebraic structures. The higher the symmetry, the stronger are the restrictions put on the possible form of correlation functions. It is known that the conformal invariance defines two-point correlation functions up to a constant amplitude and three-point correlation functions as finite linear combinations with constant coefficients of known functions (Polyakov triangles) [1–3]. For the two-dimensional conformal field theory its most fundamental features are described by the Virasoro algebra [4, 5]. A possible extension of these algebraic methods on conformal field theory (QFT) in D dimensions was suggested for arbitrary D in [6]. There it was proposed to use the algebra of the general coordinate transformation as an analogue of the Virasoro algebra for the D-dimensional case. In [6] the Green functions for operators ϕ^2, ϕ^4 were studied in the ϕ^4 theory. For a generalized diffeomorphism composed of diffeomorphism and Weyl transformation the Ward identities for these Green functions were obtained. These are similar to those used in the two-dimensional QFT. In this paper we generalize the results of [6] for all the composite operators of the scalar Euclidean D-dimensional QFT and construct the infinite algebraic structure analogous to that presented by the Virasoro algebra in two dimensions.

The popular belief [5] that for integer $D > 2$ the symmetry of QFT is finite (because conformal transformation is completely specified by $(D+1)(D+2)/2$ parameters) appears to be erroneous. For arbitrary D similar to the two-dimensional case, the global conformal transformations form a subgroup of an infinite-dimensional group of the QFT symmetry. In order to describe this symmetry we analyse the diffeomorphism and Weyl transformation of D-dimensional field theory in curved space and the most essential features of its renormalization procedure [7]. It enables us to reveal an infinite algebraic structure in the D-dimensional Euclidean QFT and to obtain its basic relations.

If the normalization factors of field operators are fixed, the correlation function in the QFT is defined by anomalous dimensions of fields and coefficients of the Wilson expansion [3, 5], and from this point of view exact analytical relations connecting these characteristics seem to be especially important. In this paper we construct the sum rules of such a kind for dimensions of fields and the Wilson coefficients for the Euclidean scalar QFT in an arbitrary dimension D. They follow from the Wilson operator product expansion and crossing symmetry for the four-point correlation function [8].

The paper is organized as follows. For eliciting an inherent algebraic structure of the Euclidean field theory we consider in a D-dimensional curved space an auxiliary model (AM) of scalar field theory with the action of the most general form. The general coordinate transformations and the local scale transformations of AM are studied in sections 2 and 3. In section 4 the infinite symmetry of the Euclidean D-dimensional renormalized field theory is determined from the symmetry of the renormalized AM. In section 5 the symmetry properties of the D-dimensional Euclidean QFT are investigated. The specific features of the Wilson operator product expansion and crossing symmetry in the QFT are considered in sections 5 and 6. The most important points of derivation of sum rules for anomalous dimensions and Wilson coefficients are presented in section 7, and technical details are described in the appendix. The results are discussed in section 8.

2. Diffeomorphism transformations

For a curved D-dimensional space with metric $\gamma_{\mu\nu}$ the general coordinate (diffeomorphism) transformations are defined in the following way. The infinitesimal reparametrization of the coordinates x is written as $\delta_\alpha^{\mathrm{DT}} x^\mu = \alpha^\mu(x)$, where $\alpha_\mu(x)$ are the parameters of transformation. The commutation relation for diffeomorphism transformations (DT) is of the form

$$\left[\delta_\alpha^{\mathrm{DT}}, \delta_\beta^{\mathrm{DT}}\right] = \delta_{[\alpha,\beta]}^{\mathrm{DT}}, \tag{1}$$

where $[\alpha, \beta] = (\alpha\nabla)\beta - (\beta\nabla)\alpha$ is the commutator of vector fields (∇_λ denotes the covariant derivative, $\nabla_\lambda\gamma_{\mu\nu} = 0$). For tensor fields,

$$\delta_\alpha^{\mathrm{DT}} F(x) = L_\alpha F(x). \tag{2}$$

Here L_α denotes the Lie derivative defined by the vector field $\alpha^\mu(x)$:

$$L_\alpha F^{\mu_1,\ldots,\mu_m}_{\nu_1,\ldots,\nu_n}(x) = (\alpha\nabla)F^{\mu_1,\ldots,\mu_m}_{\nu_1,\ldots,\nu_n} + \sum_{i=1}^{n} \nabla_{\nu_i}\alpha^{\lambda_i} F(x)^{\mu_1,\ldots,\mu_m}_{\nu_1,\ldots,\lambda_i,\ldots,\nu_n} - \sum_{i=1}^{m} \nabla_{\lambda_i}\alpha^{\mu_i} F(x)^{\mu_1,\ldots,\lambda_i,\ldots,\mu_m}_{\nu_1,\ldots,\nu_n}.$$

In particular, for the scalar field ϕ and the metric $\gamma_{\mu\nu}(x)$ we have

$$\delta_\alpha^{\mathrm{DT}}\phi(x) = (\alpha\nabla)\phi(x), \qquad \delta_\alpha^{\mathrm{DT}}\gamma_{\mu\nu} = \nabla_\mu\alpha_\nu + \nabla_\nu\alpha_\mu. \tag{3}$$

Let us introduce the notation

$$\omega_\alpha^{\mu\nu}(\gamma) \equiv \nabla^\mu\alpha^\nu + \nabla^\nu\alpha^\mu - \frac{2}{D}(\nabla\alpha)\gamma^{\mu\nu}.$$

When $\omega_\alpha^{\mu\nu} = 0$, $\alpha(x) \neq 0$, the vector $\alpha(x)$ and the corresponding transformation $\delta_\alpha^{\mathrm{conf}} \equiv \delta_\alpha^{\mathrm{DT}}$ will be called conformal. It is well known that in the flat space $\delta_\alpha^{\mathrm{conf}} x$ is the conformal transformation (CT) of the coordinates x. The commutator $[\alpha, \beta]$ of conformal vectors α, β is conformal. Therefore, it follows from (1) that the CTs (if they exist for the given metric $\gamma(x)$) form a subgroup of the DT group. This subgroup will be called conformal.

Let $\Phi(x)_{\mu_1,\ldots,\mu_n}$ denote the covariant tensor obtained by products of covariant derivatives of the field ϕ and the curvature tensors with possible contractions of a part of indices. The set $\{\boldsymbol{\Phi}\}$ of all such tensors can be used as the basis for constructing diffeomorphism invariant local functionals of the field ϕ. Thus the most general form of the diffeomorphism invariant local action of the field ϕ in the curved D-dimensional space can be written as follows [9, 10]:

$$S(A, \gamma, \phi) = \int \mathrm{d}x \sqrt{\gamma} L(\phi(x), A(x)),$$

where $\gamma \equiv \det \gamma_{\mu\nu}$, $L(\phi(x), A(x))$ is the Lagrangian:

$$L(\phi(x), A(x), \gamma(x)) = \sum_{\Phi_i \in \{\boldsymbol{\Phi}\}} A^i(x)\Phi_i(x),$$

and $A^i(x)$ denotes the contravariant tensor source corresponding to the covariant tensor field $\Phi_i(x)$. In the Lagrangian the indices of sources and the fields are contracted.

The generating functional for the connected Green functions of the Euclidean quantum field theory with action S has the form

$$W(A, \gamma) = \ln \int \exp\{-S(\phi, A)\} D\phi.$$

The metric $\gamma_{\mu\nu}(x)$ can be considered as the source for the energy–momentum tensor.

Taking into account the diffeomorphism invariance of the action $S(\phi, A)$ and using the Schwinger equations for W it is easy to show that the functional $W(A, \gamma)$ is invariant with respect to the DTs:

$$\delta^{\mathrm{DT}}_\alpha W(A, \gamma) = D^{\mathrm{DT}}_\alpha W(A, \gamma) = 0. \tag{4}$$

We have used the notation

$$D^{\mathrm{DT}}_\alpha(A, \gamma) \equiv \delta^{\mathrm{DT}}_\alpha \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} + \sum_i \delta^{\mathrm{DT}}_\alpha A^i \frac{\delta}{\delta A^i},$$

where $\delta^{\mathrm{DT}}_\alpha \gamma_{\mu\nu}$, $\delta^{\mathrm{DT}}_\alpha A^i$ are defined by (2), (3). Obviously, the operators $D^{\mathrm{DT}}_\alpha(A, \gamma)$ form a representation of the diffeomorphism algebra:

$$\left[D^{\mathrm{DT}}_\alpha(A, \gamma), D^{\mathrm{DT}}_\beta(A, \gamma)\right] = D^{\mathrm{DT}}_{[\alpha,\beta]}(A, \gamma).$$

3. Weyl transformations

We consider now the group of the Weyl transformations (WT). For the metric $\gamma_{\mu\nu}$, the infinitesimal WT is defined as the local rescaling

$$\delta^W_\sigma \gamma_{\mu\nu}(x) = -2\sigma(x)\gamma_{\mu\nu}(x) \tag{5}$$

specified by the scalar function $\sigma(x)$.

These transformations form the commutative algebra

$$\left[\delta^W_\sigma, \delta^W_\rho\right] = 0. \tag{6}$$

For the field ϕ we define the WT in the following way:

$$\delta^W_\sigma \phi(x) = \sigma(x) d_\phi \phi(x), \tag{7}$$

where $d_\phi = (D-2)/2$ is the canonical dimension of the field ϕ. Definitions (5), (7) make it possible to define the WT for the set of fields Φ:

$$\delta_\sigma^W \Phi(x) \equiv \left(\delta_\sigma^W \gamma_{\mu\nu} \frac{\delta}{\delta\gamma_{\mu\nu}} + \delta_\sigma^W \phi \frac{\delta}{\delta\phi}\right) \Phi(x).$$

This transformation can be written in the form

$$\delta_\sigma^W \Phi_i(x) = \sum_j M_i^j(\sigma)\Phi_j(x),$$

with the matrix $M_i^j(\sigma) = M_i^j(\sigma,\gamma)$ satisfying the relation

$$\delta_\sigma^W \gamma_{\mu\nu} \frac{\delta}{\delta\gamma_{\mu\nu}} M(\rho) - \delta_\rho^W \gamma_{\mu\nu} \frac{\delta}{\delta\gamma_{\mu\nu}} M(\sigma) + [M(\sigma), M(\rho)] = 0. \tag{8}$$

Let us define the operator

$$D_\sigma^W(A,\gamma,\phi) \equiv \delta_\sigma^W \gamma_{\mu\nu} \frac{\delta}{\delta\gamma_{\mu\nu}} + \delta_\sigma^W \phi \frac{\delta}{\delta\phi} + \sum_i \delta_\sigma^W A^i \frac{\delta}{\delta A^i},$$

where

$$\delta_\sigma^W A^i \equiv \sum_j \left(\sigma D \delta_j^i - M_j^i(\sigma)\right) A^j. \tag{9}$$

It can be considered as a general form of the infinitesimal WT suitable for all fields and sources because from (8), (9) it follows that

$$\left[D_\sigma^W(A,\gamma,\phi), D_\rho^W(A,\gamma,\phi)\right] = 0.$$

For the WT defined in this way one can easily prove that the action S is invariant:

$$D_\sigma^W(A,\gamma,\phi) S(A,\gamma,\phi) = 0. \tag{10}$$

Similar to the case of the DTs, it follows from (10) that the functional W is invariant with respect to the WTs:

$$D_\sigma^W(A,\gamma) W = 0, \tag{11}$$

where

$$D_\sigma^W(A,\gamma) \equiv D_\sigma^W(A,\gamma,0) = \delta_\sigma^W \gamma_{\mu\nu} \frac{\delta}{\delta\gamma_{\mu\nu}} + \sum_i \delta_\sigma^W A^i \frac{\delta}{\delta A^i}. \tag{12}$$

The operators $D_\sigma^W(A,\gamma)$ form a representation of the WT algebra:

$$\left[D_\sigma^W(A,\gamma), D_\rho^W(A,\gamma)\right] = 0.$$

For a constant σ,

$$\delta_\sigma^W \Phi_i = d_i \sigma \Phi_i, \qquad M_j^i(\sigma) = \delta_j^i \sigma d_j, \qquad \delta_\sigma^W A^i = \bar{d}_i \sigma A^i,$$

where the constant parameters $d_i = d_i(D)$, $\bar{d}_i \equiv D - d_i = \bar{d}_i(D)$ are the dimensions of the field Φ_i and the source A^i. For the field $\Phi_0 \equiv \nabla_\mu\phi\nabla^\mu\phi$ we have $d_0 = D$, and $\bar{d}_0 = 0$ for the corresponding source A^0. If the source A^i is dimensionless, i.e. $\bar{d}_i(D) = 0$ for some definite value $D = \mathcal{D}_i$ of the space dimension, the dimension $\mathcal{D}_i$ is called logarithmic for A_i. For a given $\mathcal{D}$ we denote the dimensions of fields and sources as $d_j^{\log} = d_j|_{D=\mathcal{D}}$, $\bar{d}_j^{\log} = \bar{d}_j|_{D=\mathcal{D}}$.

4. Renormalization

To perform the renormalization procedure we choose the source A that defines the logarithmic dimension $\mathcal{D}$ which is considered as a fixed parameter specifying the renormalized theory. The generating functional W_r for renormalized Green functions is defined as follows:

$$W_r(J,\gamma) \equiv W(A(J,\gamma),\gamma).$$

The functions $A(J,\gamma)$ on the right-hand side are of the form

$$A^i = \mu^{\Delta_i} F^i(J,\gamma,D). \tag{13}$$

Here μ is an auxiliary scaling parameter,

$$\Delta_i = \Delta_i(D) = \bar{d}_i(D) - \bar{d}_i^{\,\log}, \qquad \left.\frac{\partial F^i(J,\gamma,D)}{\partial J^i}\right|_{J=0} = 1.$$

The function $F(J,\gamma,D)$ obeys the homogeneity condition

$$D^{\log}(J,\gamma)F^i(J,\gamma,D) = \bar{d}_i^{\log} F^i(J,\gamma,D), \tag{14}$$

where

$$D^{\log}(J,\gamma) \equiv \sum_i \bar{d}_i^{\log} J^i \frac{\delta}{\delta J^i} - 2\gamma_{\mu\nu}\frac{\delta}{\delta\gamma_{\mu\nu}}. \tag{15}$$

It is also supposed that the functions $J(A,\gamma)$ defined by (13) are the tensors with respect to DTs:

$$\delta_\alpha^{\mathrm{DT}} J^i = L_\alpha J^i. \tag{16}$$

The operators $D_\alpha^{\mathrm{DT}}(A,\gamma)$, $D_\sigma^W(A,\gamma)$ can be presented in terms of the variables of W_r. It follows from (12), (16) that

$$D_\alpha^{\mathrm{DT}}(A,\gamma) \equiv \mathcal{D}_\alpha^{\mathrm{DT}}(J,\gamma) = D_\alpha^{\mathrm{DT}}(J,\gamma), \tag{17}$$

$$D_\sigma^W(A,\gamma) \equiv \mathcal{D}_\sigma^W(J,\gamma) = \delta_\sigma^W \gamma_{\mu\nu}\frac{\delta}{\delta\gamma_{\mu\nu}} + \sum_i \delta_\sigma^W J^i \frac{\delta}{\delta J^i}. \tag{18}$$

The WT for the sources J can be obtained from (9), (12), (13):

$$\delta_\sigma^W J^i = \sum_j T_j^i \left\{ \sum_k \left[\sigma D\delta_k^j - M_k^j(\sigma)\right]F^k - 2\sigma\gamma_{\mu\nu}\frac{\delta}{\delta\gamma_{\mu\nu}}F^j \right\}.$$

Here T_j^i is the element of the matrix T defined as follows:

$$\sum_j T_j^i \frac{\partial F^j}{\partial J^k} = \delta_k^i.$$

Since the commutation relations do not depend on the choice of variables,

$$\left[\mathcal{D}_\alpha^{\mathrm{DT}}(J,\gamma), \mathcal{D}_\beta^{\mathrm{DT}}(J,\gamma)\right] = \mathcal{D}_{[\alpha,\beta]}^{\mathrm{DT}}(J,\gamma), \tag{19}$$

$$\left[\mathcal{D}_\sigma^W(J,\gamma), \mathcal{D}_\rho^W(J,\gamma)\right] = 0. \tag{20}$$

In virtue of (13), (14), (15) we have $D^{\log}(J,\gamma) = D^{\log}(A,\gamma)$. Hence, for σ being constant we obtain the relations

$$\begin{aligned} D^W_\sigma(A,\gamma) &= \sigma D^{\log}(A,\gamma) + \sigma \sum_i \Delta_i A_i \frac{\delta}{\delta A_i} \\ &= \sigma D^{\log}(J,\gamma) + \sigma \sum_i \mu \frac{\partial A_i}{\partial \mu}\bigg|_{J=\text{const}} \frac{\delta}{\delta A_i} \\ &= \sigma D^{\log}(J,\gamma) + \sigma\mu \frac{\partial}{\partial\mu}\bigg|_{J=\text{const}} - \sigma\mu\frac{\partial}{\partial\mu}\bigg|_{A=\text{const}}, \end{aligned}$$

and

$$\delta^W_\sigma J^i = D^W_\sigma(A,\gamma)J^i = \sigma\left(\bar{d}^{\log}_i J^i - \mu \frac{\partial J^i}{\partial\mu}\bigg|_{A=\text{const}}\right). \tag{21}$$

It follows from (4), (11) that $W_r(J,\gamma)$ is invariant with respect to the diffeomorphism and Weyl transformations:

$$\mathcal{D}^{\mathrm{DT}}_\alpha(J,\gamma)W_r(J,\gamma) = 0, \tag{22}$$

$$\mathcal{D}^W_\sigma(J,\gamma)W_r(J,\gamma) = 0. \tag{23}$$

For usual models of the quantum field theory the functional $W_r(\lambda_r, J_r, J_\gamma)$ in the Euclidean D-dimensional space can be constructed from $W_r(J,\gamma)$ in the following way:

$$W_r(\lambda_r, J_r, J_\gamma) = W_r(\lambda_r + J_r, \gamma^E + J_\gamma).$$

Here λ_r, J_r denote the set of renormalized parameters and the set of the sources of renormalized composite operators, respectively. The source of the energy–momentum tensor and the metric of the D-dimensional Euclidean space are denoted by J_γ and γ^E, respectively. If nonzero $\lambda_i \neq 0$ occur only when $\bar{d}^{\log}_i \geqslant 0$, then the model is renormalizable. In this case by choosing the appropriate functions F^i in (13) the functional $W_r(\lambda_r, J_r, J_\gamma)$ and the operators $D^{\mathrm{DT}}_\alpha(\lambda_r + J_r, \gamma^E + J_\gamma)$, $D^W_\sigma(\lambda_r + J_r, \gamma^E + J_\gamma)$ are finite for $D = \mathcal{D}$, finite parameters λ_r and sources J_r, J_γ [9–11]. For $W_r(\lambda_r, J_r, J_\gamma)$ equation (23) with a constant σ appears to be the usual renormalization group equation (if one takes into account (21) and (22) with $\alpha(x) = x\sigma$).

5. Critical point

Combining the DT and the WT one can obtain the transformation

$$\delta_\alpha = \delta^{\mathrm{DT}}_\alpha + \delta^W_\sigma\big|_{\sigma=\frac{\nabla\alpha}{D}}.$$

From the commutation relations (1), (6), it follows that

$$[\delta_\alpha, \delta_\beta] = \delta_{[\alpha,\beta]}.$$

This means that the transformations δ_α form the representation of the diffeomorphism algebra. In virtue of (3), (5) we have

$$\delta_\alpha\gamma^{\mu\nu} = -\omega^{\mu\nu}_\alpha.$$

Hence, $\delta_\alpha\gamma^{\mu\nu} = 0$ for conformal α. Let us introduce the operators

$$\mathcal{D}_\alpha(J,\gamma) \equiv \mathcal{D}^{\mathrm{DT}}_\alpha(J,\gamma) + \mathcal{D}^W_\sigma(J,\gamma)\big|_{\sigma=\frac{\nabla\alpha}{D}},$$

$$\mathcal{D}^r_\alpha(\lambda_r, J_r, J_\gamma) \equiv \mathcal{D}_\alpha(\lambda_r + J_r, \gamma^E + J_\gamma).$$

It follows from (19), (20) that

$$\left[\mathcal{D}^r_\alpha(\lambda_r, J_r, J_\gamma), \mathcal{D}^r_\beta(\lambda_r, J_r, J_\gamma)\right] = \mathcal{D}^r_{[\alpha,\beta]}(\lambda_r, J_r, J_\gamma),$$

i.e. the operators D^r_α form the representation of the diffeomorphism algebra. In virtue of the diffeomorphism and Weyl invariance of W_r,

$$\mathcal{D}^r_\alpha(\lambda_r, J_r, J_\gamma) W_r(\lambda_r, J_r, J_\gamma) = 0. \tag{24}$$

If

$$\mathcal{D}^r_\alpha(\lambda^*, 0, 0) = -\omega^{\mu\nu}_\alpha(\gamma^E)\frac{\delta}{\delta\gamma^{\mu\nu}}$$

for the parameters $\lambda_r = \lambda^*$ of the renormalized Euclidean theory, we call this set of parameters the critical point. It can be proven that this equality is equivalent to relations defining the fixed point [11] in the renormalization group theory.

For conformal α let us denote $\mathcal{D}^{\text{conf}}_\alpha(J_r, J_\gamma) \equiv \mathcal{D}^r_\alpha(\lambda^*, J_r, J_\gamma)$. For the flat space, it follows from (24) that

$$\mathcal{D}^{\text{conf}}_\alpha(J_r, J_\gamma) W_r(\lambda^*, J_r, J_\gamma) = 0. \tag{25}$$

In virtue of $\mathcal{D}^{\text{conf}}_\alpha(0, 0) = 0$, equality (25) means the usual conformal invariance of the Euclidean quantum field theory at the critical point.

6. Wilson expansion

The obtained infinite number of Ward identities present the algebraic structure of QFT in terms of linear differential equations for W_r in first-order variation derivatives. The well-known Wilson asymptotic expansion is written as a differential relation including variation derivatives of second order:

$$\frac{\delta}{\delta J^i(x)}\frac{\delta}{\delta J^k(y)} W_r = \sum_l \int \mathrm{d}z\, K_{ijl}(x, y, z)\frac{\delta}{\delta J^l(z)} W_r. \tag{26}$$

It can be considered as a completing condition for the considered algebraic structure. In the QFT the Wilson-expansion series are convergent [12]; the functions $K_{ijl}(x, y, z)$ being three-point correlation functions are defined exactly up to the finite number of constant (Wilson coefficients) by dimensions of field operators. In this paper we study restrictions following from (26) for dimensions of fields and Wilson coefficients. For this purpose we introduce some definitions and notations.

Let $x^{(n)}$ be a symmetric traceless tensor of rank n constructed from components x_a, $a = 1, \dots, d$, of the D-dimensional vector x and Kronecker symbols:

$$x^{(n)}_{a_1\cdots a_n} = x_{a_1}\cdots x_{a_n} - \text{traces}.$$

By definition, the contraction of $x^{(n)}$ with $y^{(n)}$ is written as

$$x^{(n)} y^{(n)} = \sum_{k=0}^{\{\frac{n}{2}\}} d^{(n)}_k x^{2k} y^{2k} (xy)^{n-2k} \equiv \mathcal{F}^{(n)}(x^2y^2, xy),$$

where $\{n/2\}$ denotes the integer part of $n/2$, and $d^{(n)}_0 \equiv 1$. With fixing $\mathcal{F}^{(n)}(0, b) = b^n$ and condition $\partial^2_x \mathcal{F}^{(n)}(x^2y^2, xy) = 0$ the function $\mathcal{F}^{(n)}(a, b)$ is defined unambiguously:

$$\mathcal{F}^{(n)}(a, b) = b^n \sum_{k=0}^{\{\frac{n}{2}\}} d^{(n)}_k \left(\frac{a}{b^2}\right)^k, \qquad d^{(n)}_k = \frac{(-1)^k n!\Gamma(\xi + n - k - 1)}{4^k k!(n-2k)!\Gamma(\xi + n - 1)}.$$

We shall use the following notation:

$$L(\alpha; x) \equiv \frac{1}{(x^2)^\alpha}, \qquad L^{(n)}(\alpha; x) \equiv x^{(n)} L(\alpha + n; x)$$

$$\lambda_a(x; y, z) \equiv \frac{(x-y)_a}{(x-y)^2} - \frac{(x-z)_a}{(x-z)^2},$$

$$\begin{aligned} V(\alpha_1, \alpha_2, \alpha_3; x_1, x_2, x_3) &= L(\Delta_{12}; x_1 - x_2) L(\Delta_{13}; x_1 - x_3) L(\Delta_{23}; x_2 - x_3) \\ &= (x_1 - x_2)^{-2\Delta_{12}} (x_1 - x_3)^{-2\Delta_{13}} (x_2 - x_3)^{-2\Delta_{23}} \end{aligned}$$

$$V^{(n)}(\alpha, \beta, \gamma; x, y, z) = V(\alpha, \beta, \gamma; x, y, z) \lambda^{(n)}(x; y, z),$$

where x, y, z, x_1, x_2, x_3 are the vectors of the D-dimensional space, and

$$\Delta_{12} = \frac{\alpha_1 + \alpha_2 - \alpha_3}{2}, \qquad \Delta_{13} = \frac{\alpha_1 + \alpha_3 - \alpha_2}{2}, \qquad \Delta_{23} = \frac{\alpha_2 + \alpha_3 - \alpha_1}{2}.$$

The Wilson expansion for the four-point correlation function $W(x, y, s, t)$ of the scalar field Φ_α with the dimension α reads [3, 12]

$$W(x, y, s, t) = \sum_{n,l} f_{ln} \int V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, y) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, s, t)\, \mathrm{d}z. \tag{27}$$

Here, f_{ln} are the Wilson coefficients, $V^{(l)}(z, x, y; \beta_{ln}, \alpha, \alpha)$ is (up to a constant amplitude) the three-point correlation function of two fields Φ_α and one symmetric traceless one-component tensor field with the dimension d_n, and $\tilde{\beta}_{ln}$ denotes the 'shadow' with respect to the d_{nl} dimension:

$$\tilde{\beta}_{ln} \equiv 2\xi - \beta_{ln} - 2l.$$

In virtue of $V^{(l)}(z, x, y; \beta_{ln}, \alpha, \alpha) = (-1)^l V^{(l)}(z, x, y; \beta_{ln}, \alpha, \alpha)$ and symmetry $W(x, y, s, t) = W(y, x, s, t) = W(x, s, y, t)$ we conclude that in (27) the summation parameter l is even, and the crossing symmetry equation

$$\begin{aligned} &\sum_{n,l} f_{ln} \int V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, y) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, s, t)\, \mathrm{d}z \\ &\qquad = \sum_{n,l} f_{ln} \int V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, s) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, y, t)\, \mathrm{d}z \end{aligned} \tag{28}$$

must be fulfilled. It is a nontrivial restriction on the possible values of the Wilson coefficients and dimensions of fields. We show how one can be expressed in the form of exact analytical relations not containing coordinates x, y, s, t, z.

7. Crossing symmetry and sum rules

It will be convenient for compact writing of formulae to use the notation ξ for the half dimension of space D and a short notation for the product of Γ-functions:

$$\xi \equiv \frac{D}{2}, \qquad \Gamma(a, b, \ldots, c) \equiv \Gamma(a)\Gamma(b)\cdots\Gamma(c).$$

If our expressions contain the letter with prime, it will have the following meaning:

$$\alpha' \equiv \xi - \alpha.$$

Equality (28) is exact, but it is an integral equation with infinite number of terms, and a direct analysis of them is not easy. We obtain an evident form for the following consequence of the crossing symmetry equation:

$$\sum_{n,l} f_{ln} \int \frac{(s-t)^{(m)} V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, y) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, s, t)}{(s-t)^{2(\gamma+m)}} \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}z$$
$$= \sum_{n,l} f_{ln} \int \frac{(s-t)^{(m)} V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, s) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, y, t)}{(s-t)^{2(\gamma+m)}} \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}z. \tag{29}$$

It is important that (29) must be fulfilled for arbitrary γ and all integer m.

The first step of the calculation is a direct integration over x. In the appendix it is shown that with help of the formula

$$\int L^{(n)}(\alpha; x-z) L(\beta; z-y) \, \mathrm{d}z = \pi^{\xi} \frac{\Gamma(\alpha', \beta', \xi - \alpha' - \beta' + n)}{\Gamma(\alpha + n, \beta, 2\xi - \alpha - \beta)} L^{(n)}(\alpha + \beta - \xi; x - y), \tag{30}$$

one can integrate $V^{(l)}(z, x, y; \beta_{ln}, \alpha, \alpha)$ over x. After that the crossing symmetry equation takes the form

$$\sum_{n,l} f'_{ln} \int \frac{(s-t)^{(m)} (y-z)^{(l)} V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, s, t)}{(s-t)^{2(\gamma+m)} (y-z)^{2(\alpha+l-\xi)+\beta_{ln}}} \, \mathrm{d}s \, \mathrm{d}z$$
$$= \sum_{n,l} f'_{ln} \int \frac{(s-t)^{(m)} (s-z)^{(l)} V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, y, t)}{(s-t)^{2(\gamma+m)} (s-z)^{2(\alpha+l-\xi)+\beta_{ln}}} \, \mathrm{d}s \, \mathrm{d}z, \tag{31}$$

$$f'_{ln} = f_{ln} \pi^{\xi} \frac{\Gamma(\alpha' + \beta_{ln}/2 + l, -\alpha', \xi - \beta_{ln}/2)}{\Gamma(\alpha - \beta_{ln}/2, 2\xi - \alpha, \beta_{ln}/2 + l)}.$$

Now we do contractions of indices in (31). For compact writing of results we use the shift operator T_ϵ acting on functions of ϵ as follows:

$$T_\epsilon f(\epsilon) = f(\epsilon + 1).$$

Let us denote $A = (\alpha_1, \alpha_2, \alpha_3)$, $X = (x_1, x_2, x_3)$, $E = (\epsilon_1, \epsilon_2, \epsilon_3)$, $R = (\rho_1, \rho_2, \rho_3)$,

$$S(A; X) \equiv S(\alpha_1, \alpha_2, \alpha_3; x_1, x_2, x_3)$$
$$\equiv \int L(\alpha_1; x_1 - y) L(\alpha_2; x_2 - y) L(\alpha_3; x_3 - y) \, \mathrm{d}y, \tag{32}$$

$$S_n(A; X) \equiv S_n(\alpha_1, \alpha_2, \alpha_3; x_1, x_2, x_3)$$
$$\equiv \int L^{(n)}(\alpha_1; x_1 - y) \lambda^{(n)}(y; x_2, x_3) L(\alpha_2; x_2 - y) L(\alpha_3; x_3 - y) \, \mathrm{d}y. \tag{33}$$

It is shown in the appendix that the result of contraction of tensor indices in the function $S_n(A; X)$ can be presented as

$$S_n(A; X) = T^{(n)}(E, R) G(A, E, R) S(A + R; X)|_{E=R=0}, \tag{34}$$

where

$$T^{(n)}(E, R) = \mathcal{F}^{(n)}(\mathcal{M}(E, R), \mathcal{N}(E, R)),$$
$$\mathcal{M}(E, R) = T_{\epsilon_1}^2 T_{\rho_1} \left[\left(T_{\epsilon_2} + T_{\epsilon_3} \right) \left(T_{\epsilon_2} T_{\rho_2} + T_{\epsilon_3} T_{\rho_3} \right) - T_{\epsilon_2} T_{\epsilon_3} T_{\rho_1} \right],$$
$$\mathcal{N}(E, R) = \tfrac{1}{2} T_{\epsilon_1} \left[\left(T_{\epsilon_2} + T_{\epsilon_3} \right) \left(T_{\rho_2} - T_{\rho_3} \right) + \left(T_{\epsilon_2} - T_{\epsilon_3} \right) T_{\rho_1} \right].$$
$$G(A, E, R,) = \prod_{i=1}^{3} \frac{\Gamma(\alpha_{nli} + \rho_i, 1 - \alpha'_{nli} + \rho_i)}{\Gamma(\alpha_{nli} + \epsilon_i, 1 - \alpha'_{nli})},$$
$$\alpha_{nl1} = \alpha + \frac{\beta_{nl}}{2} - \xi, \qquad \alpha_{nl2} = \alpha_{nl3} = \xi - \frac{\beta_{nl}}{2} - l.$$

With (34) the problem of integration in (31) is reduced to the case $l = 0$ and is solved directly by integration formula (30) (see the appendix). The final result is formulated in the following way. The crossing symmetry relation (31) is equivalent to the equality

$$\sum_{nl} f'_{nl} \Psi_{nl}(\alpha, \beta_{nl}; \gamma, m) = 0 \tag{35}$$

fulfilling for an arbitrary value of the parameter γ and integer $m \geqslant 0$. The function $\Psi_{nl}(\alpha, \beta_{nl}; \gamma, m)$ can be written in the form

$$\Psi_{nl}(\alpha, \beta_{nl}; \gamma, m) = T^{(n)}(E, R) Q(\alpha, \beta_{nl}, E, R) \Omega_m(\alpha, \beta_{nl}, \gamma, R)|_{E=R=0},$$

where

$$Q(\alpha, \beta_{nl}, E, R) = \frac{\pi^{2(\xi+1)}(-1)^{\rho_1+\rho_2}\Gamma(\alpha_{nl2}+\rho_3, 1-\alpha'_{nl2}+\rho_3)}{\sin(\pi\alpha'_{nl1})\sin(\pi\alpha'_{nl2})\prod_{i=1}^{3}\Gamma(\alpha_{nli}+\epsilon_i, 1-\alpha'_{nli})},$$

$$\Omega_m(\alpha, \beta_{nl}, \gamma, R) = \frac{\Gamma(\sigma'_1, \sigma'_2, \sigma'_3+m, \sigma'_4+m)}{\Gamma(\sigma_1+m, \sigma_2+m, \sigma_3, \sigma_4)} - \frac{\Gamma(\tau'_1, \tau'_2, \tau'_3+m, \tau'_4+m)}{\Gamma(\tau_1+m, \tau_2+m, \tau_3, \tau_4)},$$

with

$$\begin{aligned}
&\sigma_1 = \gamma + \alpha + \frac{\beta_{nl}}{2} - \xi + l, && \sigma_2 = \gamma + \alpha - \frac{\beta_{nl}}{2} - l + \rho_2 + \rho_3,\\
&\sigma_3 = 2\xi - \alpha - \rho_2 - \gamma, && \sigma_4 = 3\xi + l - 2\alpha - \rho_1 - \rho_2 - \rho_3 - \gamma,\\
&\tau_1 = \gamma, && \tau_2 = \alpha - l + \rho_1 + \rho_3 + \gamma - \xi,\\
&\tau_3 = 3\xi - \frac{\beta_{nl}}{2} - \alpha - \rho_1 - \gamma,\\
&\tau_4 = 2\xi + 2l + \frac{\beta_{nl}}{2} - \alpha - \rho_1 - \rho_2 - \rho_3 - \gamma.
\end{aligned}$$

8. Conclusion

It has been shown that for scalar Euclidean field theories at the critical point $\lambda_r = \lambda^*$ the operators $\mathcal{D}^r_\alpha(\lambda_r, J_r, J_\gamma)$ represent the generators of the DTs. The functional $W_r(\lambda_r, J_r, J_\gamma)$ is invariant with respect to the infinite set of the DTs defined by $\mathcal{D}^r_\alpha(\lambda_r, J_r, J_\gamma)$, including conformal transformations of the D-dimensional Euclidean space. The Ward identities (24), (25) are the formal expressions of this invariance.

The Weyl invariance is described by equation (23), where the WT is presented by the differential operator $\mathcal{D}^W_\sigma(J, \gamma)$. For a constant σ it generates the usual renormalization group equation (if one puts in (24) $\alpha(x) = x\sigma$). With arbitrary $\sigma(x)$ relation (23) could be regarded as a solution of the problem of local generalization for the renormalization group equations [14].

For the constructed algebraic structure the Wilson expansion (26) is included as an additional relation. It was used to derive the sum rule (35) which must be true for arbitrary λ and integer m. This nontrivial condition enables us to hope that these sum rules contain an essential information about the dimensions of composite operators and the Wilson coefficients.

We have obtained the following result. There exists an infinite-dimensional algebraic structure corresponding to each model of the D-dimensional Euclidean scalar QFT. It is described by the commutation relations (19), (20), Wilson expansion formula (26) and Ward identities (23)–(25). By choosing the logarithmic dimension $\mathcal{D}$ defining dimensions of sources $\bar{d}_i^{\log}$, the concrete scalar field theory model is fixed. Additional restrictions follow from the sum rules (35).

From the physical point of view, we revealed a hidden infinite-dimensional symmetry in usual scalar field theory at the critical point. It generates an infinite number of the Ward identities being an analogue to the infinite number of conservation laws in completely integrable systems. The critical exponents and Wilson coefficients of the model obey the sum rules which are written as exact transcendental equations in terms of Γ-functions. It enables us to conclude that investigation of the considered infinite-dimensional algebraic structures is important for the deeper understanding of inherent features of the critical state.

It can also be useful for the practical calculation of the characteristic of the critical behaviour in concrete models. For example, in the case of φ^4-theory we have to choose in (13) the logarithmic dimension $\mathcal{D} = 4$. If we denote $\varepsilon = (D-4)/2$ and present the right-hand side of (13) as a power series in ε^{-1}, the coefficients of this series can be interpreted in terms of renormalization constants for field operators in the minimal subtraction scheme. Then the algebraic structure of theory, presented by (19), (20), provides infinite number of algebraic relations for renormalization constants. Together with the sum rules (35) they can be used for construction of effective calculation methods for critical exponents and Wilson coefficients in the framework of the ε-expansion.

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Appendix. Details of calculations

We present some technical aspects of calculation methods used in this paper. The basic formula for our integration over coordinates of the D-dimensional space is

$$\begin{aligned} &\int L(\alpha; x-z)L(\beta; z-y)\,\mathrm{d}z = v(\alpha,\beta,\gamma)L(\gamma'; x-y), \\ &\gamma = 2\xi - \alpha - \beta,\ v(\alpha,\beta,\gamma) = \pi^{\xi}\frac{\Gamma(\alpha',\beta',\gamma')}{\Gamma(\alpha,\beta,\gamma)}. \end{aligned} \tag{A.1}$$

It can be easily proven with the help of Fourier transformation [13]. For derivatives we have

$$\begin{aligned} L^{(n)}(\alpha; x) &= \frac{x^{(n)}}{x^{2(\alpha+n)}} = \frac{(-1)^n\Gamma(\alpha)}{2^n\Gamma(\alpha+n)}\partial_x^{(n)}\frac{1}{x^{2\alpha}} \\ &= \left(-\frac{T_\epsilon\partial_x}{2}\right)^{(n)}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\epsilon)x^{2\alpha}}\bigg|_{\epsilon=0} = \left(-\frac{T_\epsilon\partial_x}{2}\right)^{(n)}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\epsilon)}L(\alpha; x)\bigg|_{\epsilon=0}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} (\partial_x^2)^n L(\alpha; x) = (\partial_x^2)^n\frac{1}{x^{2\alpha}} &= 4^n\frac{\Gamma(\alpha+n, n+1-\alpha')}{\Gamma(\alpha, 1-\alpha')x^{2(\alpha+n)}} \\ &= (4T_\rho)^n\frac{\Gamma(\alpha+\rho,\rho+1-\alpha')}{\Gamma(\alpha,1-\alpha')}L(\alpha+\rho; x)\bigg|_{\rho=0}. \end{aligned} \tag{A.3}$$

From (A.1), (A.2) we obtain the generalization of (A.1):

$$\begin{aligned} &\int L^{(n)}(\alpha; x-z)L(\beta; z-y)\,\mathrm{d}z = v^{(n)}(\alpha,\beta,\gamma)L^{(n)}(\gamma'; x-y), \\ &v^{(n)}(\alpha,\beta,\gamma) \equiv \pi^{\xi}\frac{\Gamma(\alpha',\beta',\gamma'+n)}{\Gamma(\alpha+n,\beta,\gamma)}. \end{aligned} \tag{A.4}$$

For the calculation of the integral over x in (29) we use the inversion operator R acting on the function of the D-dimensional vectors and defined as

$$Rx \equiv \frac{x}{x^2}, \qquad Rf(x, y, \ldots, z) \equiv f(Rx, Ry, \ldots, Rz).$$

The inversion operator has the following properties:

$$R^2 = 1, \qquad R\frac{1}{x^{2\alpha}} = x^{2\alpha}, \qquad R\frac{1}{(x-y)^{2\alpha}} = \frac{x^{2\alpha}y^{2\alpha}}{(x-y)^{2\alpha}},$$

$$R\lambda(0; y, z) = y - z, \qquad \det\left(\frac{\partial(Rx)}{\partial x}\right) = \frac{1}{x^{2D}}.$$

Therefore, we obtain

$$\begin{aligned}
\int V^{(l)}(0, x, y; \beta, \alpha, \alpha)\,\mathrm{d}x &= R^2 \int V^{(l)}(0, x, y; \beta, \alpha, \alpha)\,\mathrm{d}x \\
&= R\int (x-y)^{(l)} V(0, Rx, Ry; \beta, \alpha, \alpha)\,\mathrm{d}(Rx) \\
&= R\int \frac{(x-y)^{(l)}}{x^{2(d-\alpha)}y^{-2\alpha}(x-y)^{2\alpha-\beta}}\,\mathrm{d}x \\
&= \frac{v^{(l)}(\alpha - \beta/2 - l, 2\xi - \alpha, \beta/2 + l)y^{(l)}}{y^{2(\alpha-\xi)+\beta}}.
\end{aligned}$$

Hence,

$$\int V^{(l)}(z, x, y; \beta, \alpha, \alpha,)\,\mathrm{d}x = v^{(l)}(\alpha - \beta/2 - l, 2\xi - \alpha, \beta/2 + l)L^{(l)}\left(\alpha - \xi + \frac{\beta}{2}; y - z\right).$$

Thus, after integration over x in (29) one obtains (31).

It follows from the definition of $\lambda(x; y, z)_\mu$ and (A.2) that

$$\frac{\lambda(x; y, z)_\mu}{(x-y)^{2\alpha}(x-z)^{2\beta}} = D_\mu(y, z; \epsilon_1, \epsilon_2)\frac{\Gamma(\alpha, \beta)}{\Gamma(\alpha+\epsilon_1, \beta+\epsilon_2)(x-y)^{2\alpha}(x-z)^{2\beta}}\bigg|_{\epsilon_1=\epsilon_2=0},$$

and

$$\frac{\lambda^{(n)}(x; y, z)}{(x-y)^{2\alpha}(x-z)^{2\beta}} = D^{(n)}(y, z; \epsilon_1, \epsilon_2)\frac{\Gamma(\alpha, \beta)}{\Gamma(\alpha+\epsilon_1, \beta+\epsilon_2)(x-y)^{2\alpha}(x-z)^{2\beta}}\bigg|_{\epsilon_1=\epsilon_2=0},$$

where

$$D_\mu(y, z; \epsilon_1, \epsilon_2) \equiv \frac{1}{2}\left(T_{\epsilon_1}\frac{\partial}{\partial y_\mu} - T_{\epsilon_2}\frac{\partial}{\partial z_\mu}\right).$$

Using (A.2) and notations (32), (33) we obtain the following equality:

$$S_n(A; X) = \left(-\frac{T_{\epsilon_1}\partial_{x_1}}{2}\right)^{(n)} D^{(n)}(x_2, x_3; \epsilon_2, \epsilon_3)G(A, E)S(A; X)|_{E=0},$$

$$G(A, E) = \frac{\Gamma(\alpha_1, \alpha_2, \alpha_3)}{\Gamma(\alpha_1+\epsilon_1, \alpha_2+\epsilon_2, \alpha_3+\epsilon_3)}.$$

The function $S(A; X)$ is invariant with respect to translations, i.e. $S(A; x_1, x_2, x_3) = S(A; x_1 + y, x_2 + y, x_3 + y)$. Therefore

$$\left(\partial_{x_1} + \partial_{x_2} + \partial_{x_3}\right)S(A; X) = 0, \partial_{x_i}\partial_{x_j}S(A; X) = \tfrac{1}{2}\left(\partial^2_{x_k} - \partial^2_{x_i} - \partial^2_{x_j}\right)S(A; X),$$

where $i, j, k = 1, 2, 3$ and $i \neq j, i \neq k, j \neq k$. By means of (A.3), we obtain

$$\left(\frac{T_{\epsilon_1}\partial_{x_1}}{2}\right)^{2k} D^{2k}(x_2, x_3; \epsilon_2, \epsilon_3) G(A, E) S(A; X)|_{E=0}$$
$$= \frac{1}{4^{2k}} T_{\epsilon_1}^{2k} \partial_{x_1}^{2k} \left(T_{\epsilon_2}^2 \partial_{x_2}^2 + T_{\epsilon_3}^2 \partial_{x_3}^2 - 2T_{\epsilon_2} T_{\epsilon_3} \partial_{x_2}\partial_{x_3}\right)^k G(A, E) S(A; X)|_{E=0}$$
$$= \mathcal{M}(E, R)^k G(A, E, R) S(A + R; X)|_{E=0, R=0},$$

where $R = (\rho_1, \rho_2, \rho_2)$,

$$\mathcal{M}(E, R) = T_{\epsilon_1}^2 T_{\rho_1}\left[\left(T_{\epsilon_2} + T_{\epsilon_3}\right)\left(T_{\epsilon_2} T_{\rho_2} + T_{\epsilon_3} T_{\rho_3}\right) - T_{\epsilon_2} T_{\epsilon_3} T_{\rho_1}\right],$$
$$G(A, E, R,) = \prod_{i=1}^{3} \frac{\Gamma(\alpha_i + \rho_i, 1 - \alpha_i' + \rho_i)}{\Gamma(\alpha_i + \epsilon_i, 1 - \alpha_i')}.$$

Analogously we obtain the relation

$$\left(-\frac{T_{\epsilon_1}\partial_{x_1}}{2} D(x_2, x_3; \epsilon_2, \epsilon_3)\right)^l G(A, E) S(A; X)|_{E=0}$$
$$= \frac{1}{4^l} T_{\epsilon_1}^l \left(T_{\epsilon_3}\partial_{x_1}\partial_{x_3} - T_{\epsilon_2}\partial_{x_1}\partial_{x_2}\right)^l G(A, E) S(A; X)|_{E=0}$$
$$= \mathcal{N}(E, R)^l G(A, E, R) S(A + R; X)|_{E=0, R=0},$$

where

$$\mathcal{N}(E, R) = \tfrac{1}{2} T_{\epsilon_1}\left[\left(T_{\epsilon_2} + T_{\epsilon_3}\right)\left(T_{\rho_2} - T_{\rho_3}\right) + \left(T_{\epsilon_2} - T_{\epsilon_3}\right) T_{\rho_1}\right].$$

Thus, we have shown that the result of contraction of tensor indices in the function $S_n(A; X)$ can be presented as

$$S_n(A; X) = T^{(n)}(E, R) G(A, E, R) S(A + R; X)|_{E=R=0},$$

where

$$T^{(n)}(E, R) = \mathcal{F}^{(n)}(\mathcal{M}(E, R), \mathcal{N}(E, R)).$$

We can write the crossing symmetry equation (31) as

$$\sum_{n,l} f_{nl}' \int S_n(\alpha_{nl1}, \alpha_{nl2}, \alpha_{nl3}; y, s, t) L^{(m)}(\zeta_{nl} + \gamma; s - t)\, \mathrm{d}s$$
$$= \sum_{n,l} f_{nl}' \int S_n(\alpha_{nl1}, \alpha_{nl2}, \alpha_{nl3}; s, y, t) L^{(m)}(\gamma; s - t) L(\zeta_{nl}; y - t)\, \mathrm{d}s,$$

with

$$\alpha_{nl1} = \frac{\beta_{nl}}{2} - \alpha', \qquad \alpha_{nl2} = \alpha_{nl3} = \frac{\tilde{\beta}_{nl}}{2}, \qquad \zeta_{nl} = \alpha - \frac{\tilde{\beta}_{nl}}{2}.$$

By means of (A.4) we obtain

$$\int S(\alpha_1, \alpha_2, \alpha_3; y, s, t) L^{(m)}(\zeta + \gamma; s - t)\, \mathrm{d}s$$
$$= \Phi_1(\alpha_1, \alpha_2, \alpha_3, \zeta, \gamma, m) L^{(m)}(\alpha_1 + \alpha_2 + \alpha_3 + \zeta + \gamma - 2\xi; t - y),$$
$$\int S(\alpha_1, \alpha_2, \alpha_3; s, y, t) L^{(m)}(\gamma; s - t) L(\zeta; s - y)\, \mathrm{d}s$$
$$= \Phi_2(\alpha_1, \alpha_2, \alpha_3, \zeta, \gamma, m) L^{(m)}(\alpha_1 + \alpha_2 + \alpha_3 + \zeta + \gamma - 2\xi; t - y),$$

where

$$\Phi_1(\alpha_1,\alpha_2,\alpha_3,\zeta,\gamma,m) = v_m(\gamma+\zeta,\alpha_2,2\xi-\alpha_2-\gamma-\zeta)$$
$$\times\, v_m(\gamma+\zeta+\alpha_2+\alpha_3-\xi,\alpha_1,3\xi-\alpha_1-\alpha_2-\alpha_3-\gamma-\zeta),$$
$$\Phi_2(\alpha_1,\alpha_2,\alpha_3,\zeta,\gamma,m) = v_m(\gamma,\alpha_1,2\xi-\alpha_1-\gamma)$$
$$\times\, v_m(\alpha_1+\alpha_2+\gamma-\xi,\alpha_3,3\xi-\alpha_1-\alpha_2-\alpha_3-\gamma).$$

Let us denote

$$\Phi(A,\zeta,\gamma,m) = \Phi(\alpha_1,\alpha_2,\alpha_3,\zeta,\gamma,m) \equiv \Phi_1(A,\zeta,\gamma,m) - \Phi_2(A,\zeta,\gamma,m),$$
$$\Psi_{nl}(\alpha,\beta_{nl};\gamma,m) = T^{(n)}(E,R)G(A_{nl},E,R)\Phi(A_{nl}+R,\gamma,\zeta_{nl})|_{E=R=0},$$

where $A_{nl}\equiv(\alpha_{nl1},\alpha_{nl2},\alpha_{nl3})$. Equation (31) reads

$$\sum_{nl} f'_{nl}\Psi_{nl}(\alpha,\beta_{nl};\gamma,m) = 0.$$

It is fulfilled for an arbitrary value of the parameter γ and integer $m\geqslant 0$. Using notations $\bar{E}\equiv(\epsilon_1,\epsilon_3,\epsilon_2)$, $\bar{R}\equiv(\rho_1,\rho_3,\rho_2)$,

$$\mathcal{A}_m(\nu_1,\nu_2;\nu_3,\nu_4) \equiv \frac{\Gamma(\nu'_1,\nu'_2,\nu'_3+m,\nu'_4+m)}{\Gamma(\nu_1+m,\nu_2+m,\nu_3,\nu_4)},$$

we write

$$\Phi_1(A_{nl}+R,\gamma,m) = \pi^{2\xi}\prod_{i=1}^{2}\frac{\Gamma(\alpha'_{nli}-\rho_i,)}{\Gamma(\alpha_{nli}+\rho_i)}\mathcal{A}_m(\sigma_{nl1},\sigma_{nl2};\sigma_{nl3},\sigma_{nl4}),$$
$$\sigma_{nl1} = \gamma+\alpha+\frac{\beta_{nl}}{2}-\xi+l, \qquad \sigma_{nl2} = \gamma+\alpha-\frac{\beta_{nl}}{2}-l+\rho_2+\rho_3,$$
$$\sigma_{nl3} = 2\xi-\alpha-\rho_2-\gamma, \qquad \sigma_{nl4} = 3\xi+l-2\alpha-\rho_1-\rho_2-\rho_3-\gamma,$$
$$\Phi_2(A_{nl}+\bar{R},\gamma,m) = \pi^{2\xi}\prod_{i=1}^{2}\frac{\Gamma(\alpha'_{nli}-\rho_i,)}{\Gamma(\alpha_{nli}+\rho_i)}\mathcal{A}_m(\tau_{nl1},\tau_{nl2};\tau_{nl3},\tau_{nl4}),$$
$$\tau_{nl1} = \gamma, \qquad \tau_{nl2} = \alpha-l+\rho_1+\rho_3+\gamma-\xi,$$
$$\tau_{nl3} = 3\xi-\frac{\beta_{nl}}{2}-\alpha-\rho_1-\gamma,$$
$$\tau_{nl4} = 2\xi+2l+\frac{\beta_{nl}}{2}-\alpha-\rho_1-\rho_2-\rho_3-\gamma.$$

Taking into account that $\Gamma(1-x,x)=\pi/\sin(\pi x)$, we obtain

$$G(A,E,R,)\prod_{i=1}^{2}\frac{\Gamma(\alpha'_{nli}-\rho_i,)}{\Gamma(\alpha_{nli}+\rho_i)}$$
$$= \prod_{i=1}^{2}\frac{\Gamma(\alpha_{nli}+\rho_i,1-\alpha'_{nli}+\rho_i,\alpha'_{nli}-\rho_i)}{\Gamma(\alpha_{nli}+\epsilon_i,1-\alpha'_{nli},\alpha_{nli}+\rho_i)}\frac{\Gamma(\alpha_{nl2}+\rho_3,1-\alpha'_{nl2}+\rho_3)}{\Gamma(\alpha_{nl2}+\epsilon_3,1-\alpha'_{nl2})}$$
$$= \pi^2\frac{\Gamma(\alpha_{nl2}+\rho_3,1-\alpha'_{nl2}+\rho_3)}{\sin(\pi(\alpha'_{nl1}-\rho_1))\sin(\pi(\alpha'_{nl2}-\rho_2))}\prod_{i=1}^{3}\frac{1}{\Gamma(\alpha_{nli}+\epsilon_i,1-\alpha'_{nli})}.$$

For integer m, and for even n

$$\sin(\alpha+\pi m) = (-1)^m\sin(\alpha),$$
$$T^{(n)}(E,R)G(A_{nl},E,R)\Phi_2(A_{nl}+R,\gamma,\zeta_{nl})|_{E=R=0}$$
$$= T^{(n)}(E,R)G(A_{nl},\bar{E},\bar{R})\Phi_2(A_{nl}+\bar{R},\gamma,\zeta_{nl})|_{E=R=0}$$
$$= T^{(n)}(E,R)G(A_{nl},E,R)\Phi_2(A_{nl}+\bar{R},\gamma,\zeta_{nl})|_{E=R=0}.$$

Therefore we can present the function $\Psi_{nl}(\alpha, \beta_{nl}; \gamma, m)$ as follows:

$$\Psi_{nl}(\alpha, \beta_{nl}; \gamma, m) = T^{(n)}(E, R)Q(\alpha, \beta_{nl}, E, R)\Omega_m(\alpha, \beta_{nl}, \gamma, R)|_{E=R=0},$$

where

$$Q(\alpha, \beta_{nl}, E, R) = \frac{\pi^{2(\xi+1)}(-1)^{\rho_1+\rho_2}\Gamma(\alpha_{nl2} + \rho_3, 1 - \alpha'_{nl2} + \rho_3)}{\sin(\pi\alpha'_{nl1})\sin(\pi\alpha'_{nl2})\prod_{i=1}^{3}\Gamma(\alpha_{nli} + \epsilon_i, 1 - \alpha'_{nli})},$$

$$\Omega_m(\alpha, \beta_{nl}, \gamma, R) = \mathcal{A}_m(\sigma_{nl1}, \sigma_{nl2}; \sigma_{nl3}, \sigma_{nl4}) - \mathcal{A}_m(\tau_{nl1}, \tau_{nl2}; \tau_{nl3}, \tau_{nl4}).$$

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